Université Paris Dauphine

2025-2026

Introduction to Time series

TD2 - Filtering

Exercice 1 Show that the autocorrelation function of a linear process is summable, in particular a linear process is decorrelated at infinity.

Exercice 2 (Weak filtering) Suppose (Z_t) is a standardized white noise (i.e. $\mathbb{E}[Z_t] = 0, \mathbb{V}ar[Z_t] = 1$) and $(a_n) \in \ell^2(\mathbb{Z})$. We want to define the process X by

$$X_t = \sum_{k\in\mathbb{Z}} a_k Z_{t-k} \qquad \forall t \in \mathbb{Z}.$$

1. Explain why the filtering theorem does not work here.
2. Nevertheless show that, for all $t \in \mathbb{Z}$, the series $\sum_{k\in\mathbb{Z}} a_k Z_{t-k}$ converges in L^2. We call X_t its limit.
3. Show further that the process (X_t) is stationary.

Exercice 3 (Auto-regressive equation) Let $\phi \in \mathbb{R}^*$ and $Z = (Z_t)$ be a standardized white noise. We are interested in the stochastic processes $X = (X_t)$, solutions of the following auto-regressive equation :

$$X_t = \phi X_{t-1} + Z_t, \qquad t \in \mathbb{Z}.$$

1. Show that, when $|\phi| < 1$, the equation admits a unique stationary solution. Is it causal ? (i.e., Does it verify $X_t \in \overline{\text{Vect}(Z_t, Z_{t-1}, Z_{t-2}, \dots)}$ for all $t \in \mathbb{Z}$? The closure taking place in L^2).
2. Same question when $|\phi| > 1$.
3. Conversely, if $\phi = \pm 1$, show that the equation does not admit a stationary solution.
4. More generally, if $a_1, \dots, a_n$ is a sequence of real numbers satisfying

$$\sum_{i=1}^{n} a_i = 1 \qquad \text{or} \qquad \sum_{i=1}^{n} (-1)^i a_i = 1,$$

show that the auto-regressive equation

$$X_t = \sum_{i=1}^{n} a_i X_{t-i} + Z_t, \qquad t \in \mathbb{Z}$$

does not admit a stationary solution.

Exercice 4 (Invertibility) In each following case, compute the inverse of the filter $\alpha \in \ell^1(\mathbb{Z})$ if it exists.

1. $\alpha_0 = 2$, $\alpha_1 = -1$, and $\alpha_k = 0$ if $k \notin \{0, 1\}$.

2. $\alpha_0 = 1$, $\alpha_1 = 2$, and $\alpha_k = 0$ if $k \notin \{0, 1\}$.

3. $\alpha_0 = 1$, $\alpha_1 = -1$, and $\alpha_k = 0$ if $k \notin \{0, 1\}$.

Exercice 5 (Abstract version of the filtering theorem) We denote by $\ell^1(\mathbb{Z})$ the space of real and (absolutely) summable sequences, with the norm $\|\alpha\|_1 := \sum_{k\in\mathbb{Z}} |\alpha_n|$. Let E be the space of processes (X_t) bounded in L^2 equipped with the norm $\|X\|_E = \sup_{t\in\mathbb{Z}} \|X_t\|_2$. We denote by $\mathcal{L}(E)$ the space of continuous linear maps from E to E. Let us admit that E and $\mathcal{L}(E)$ equipped with their respective norms are both Banach spaces.

Define the lag operator (or backshift operator) $B \in \mathcal{L}(E)$ by $BX = (X_{t-1})_{t\in\mathbb{Z}}$.

1. Show that B is an isometric operator on E, i.e. B is invertible linear operator and satisfies $\|BX\|_E = \|X\|_E$ for all X of E.

2. Deduce that if $\alpha \in \ell^1(\mathbb{Z})$, then the series $\sum_{n\in\mathbb{Z}} \alpha_n B^n$ converges in $\mathcal{L}(E)$. We note $\phi(\alpha)$ this sum.

3. Show that $\phi(\alpha \star \beta) = \phi(\alpha) \circ \phi(\beta)$ for all $\alpha, \beta \in \ell^1(\mathbb{Z})$ (ϕ is said to be an algebraic morphism). If α is invertible, deduce that $\phi(\alpha)$ is also.

4. Show that ϕ is injective.
Hint : We can start by showing that given a white noise $(Z_t)_{t\in\mathbb{Z}}$, the map
$$\alpha \longrightarrow \sum_{n\in\mathbb{Z}} \alpha_n B^n Z \in E$$
is injective.